Computer Algorithmen für Vektorgeometrie

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Motivation

- Vector geometry is constructive, many interesting problems, nice algorithms
- Classical books on vector geometry don't use computers
- Using computers is a good training for
  - vector geometry (full understanding of concepts necessary)
  - programming exercises (implement small nice algorithms)
- New algorithms can be developed and applied (computers are not restricted to only use algorithms which are suited for hand computations)

Rotations (Givensrotations), not suited for hand-computations!

$$G_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad G_2 = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

G_1	G_2	G_3
rotation around x_3 in x_1x_2-plane	rotation around x_2 in x_1x_3-plane	rotation around x_1 in x_2x_3-plane

- $G_3G_2G_1 \begin{pmatrix} 2 & -2 & 0 \\ 4 & -6 & -1 \\ 8 & 4 & -6 \end{pmatrix} = \begin{pmatrix} -9.17 & -0.44 & 5.67 \\ 0 & -7.47 & 2.08 \\ 0 & 0 & 0.70 \end{pmatrix}$

  Rotate column vectors to upper triangular matrix (Givens Reduction)

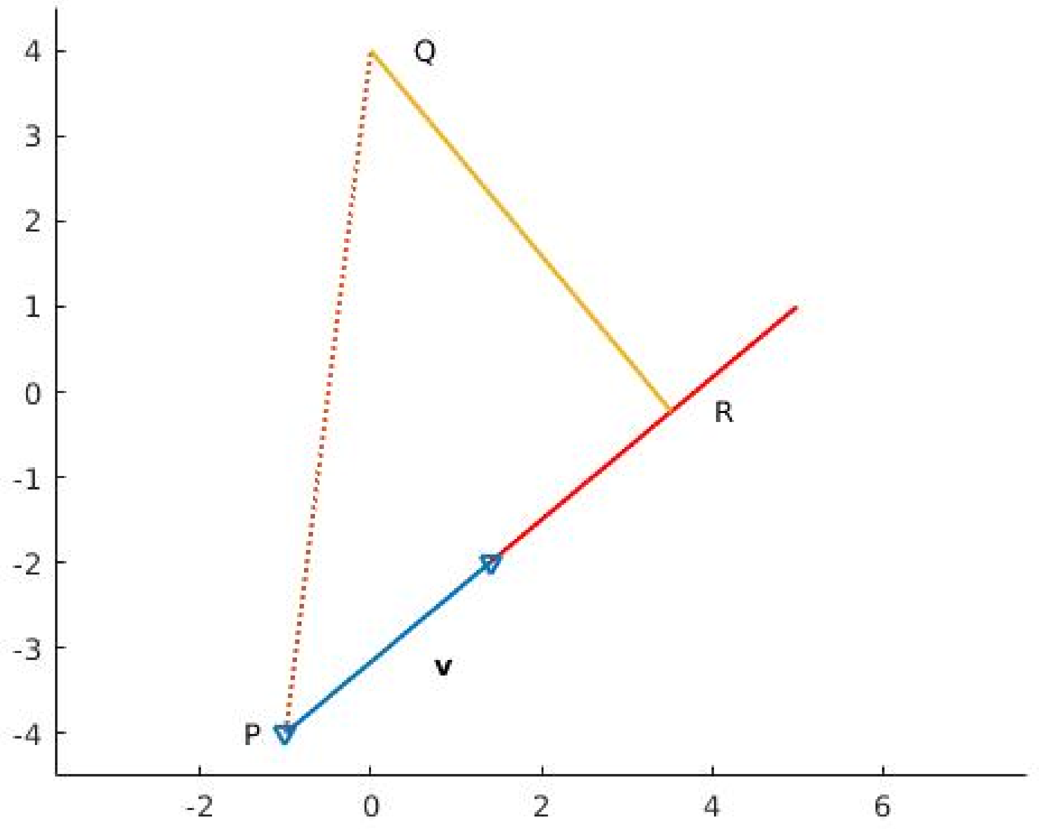
- Remarks about Descriptive Geometry!

Program for Givens-Reduction of a Linear System

```
function [R,c]=GivensReduction(A,b)
% GIVENSREDUCTION reduces the linear system A x= b to
% upper triangular form R x = c
[m,n]=size(A); [m,p]=size(b);
R=[A,b];                        % append right hand sides
for i=1:n                       % for all columns
  for k=i+1:m                   % rotate R(k,i) to 0
    if R(k,i)~=0                % skip if already 0
      cot=-R(i,i)/R(k,i);
      si=1/sqrt(1+cot^2); co=si*cot;
      G=[co,-si;si,co];         % Givens rotation matrix
      R(i:k-i:k,i:n+p)=G*R(i:k-i:k,i:n+p);
    end                         % apply to rows i and k
  end;
end
c=R(:,n+1:n+p); R=R(:,1:n);
```

Point and Straight Line

- Given point $\mathbf{Q}$ and straight line $g:\ \mathbf{X} = \mathbf{P} + \lambda\mathbf{v}$
- Compute projected point R and distance
- Solve for 2D and 3D



Point and Straight Line in 2D, Traditional Solution

```
function [d,R]=PointLine2D(Q,P,v)
 % POINTLINE2D computes the distance of point Q from the
 %            line X=P+lam*v by projecting on the normal
 %            and the projected point R
v=v/norm(v);          % unit vector
n=[v(2);-v(1)];       % construct normal
d=abs(n'*(Q-P));      % project on normal
A=[n,-v];             % R=intersect lines Q+nu n with P+ lam v
h=A'*(P-Q);           % A is orthogonal so inv(A)=A^T
R=P+h(2)*v;           % h=[nu;lam]
```

Point and Straight Line in 3D, Traditional Solution

- Construct orthogonal plane to g through $\mathbf{Q}$: $\mathbf{n} = \mathbf{v}$,
  $\implies n_1x_1 + n_2x_2 + n_3x_3 + c = 0$
- insert point $\mathbf{Q}$ to get $c = -\mathbf{Q}^\top \mathbf{n}$,
- intersect plane with line $g \implies \mathbf{R}$ projected point
- distance$= \|\mathbf{Q} - \mathbf{R}\|$

```
function [d,R]=PointLineConv(Q,P,v)
% POINTLINECONV computes the distance d and the projected
%   point R of a point Q from the straight line g: X=P+lambda v.
n=v;                       % normal of orth. plane to g
c=-Q'*n;                   % insert Q to get general normal form of plane
                           % n_1x_1+n_2x_2+n_3x_3+c=0
lam=-(n'*P+c)/(n'*v);      % cut plane with g
R=P+lam*v;                 % intersection  point R
d=norm(Q-R);               % compute distance
```

Point and Straight Line, new Algorithm

- $\mathbf{Q}$ on $g \iff \mathbf{Q} = \mathbf{P} + \lambda\mathbf{v} \iff \lambda\mathbf{v} = \mathbf{Q} - \mathbf{P}$
- $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \lambda = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$ Givensreduction $\Longrightarrow \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} \lambda = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$
- $\lambda = c_1/b_1$, projected point $\mathbf{R} = \mathbf{P} + \lambda\mathbf{v}$. Distance $d = \sqrt{c_2^2 + c_3^2}$

```
function [d,R]=PointLine(Q,P,v)
% POINTLINE computes the distance d and the projected
%   point R of a point Q from the straight line g: X=P+lambda v.
[b,c]=GivensReduction(v,Q-P);
lambda=c(1)/b(1);                 % solve for lambda
R=P+lambda*v;                     % projected point
d=norm(c(2:end));                 % distance
```

- The function `PointLine` works without changes also for 2D!

Straight Line: Conversion Parametric $\rightarrow$ Normal Form

- $\mathbf{X} = \mathbf{P} + \lambda \mathbf{v} \quad \rightarrow \quad n_1 x_1 + n_2 x_2 + c = 0$
- $\mathbf{n} = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} \implies n_1 x_1 + n_2 x_2 + c = 0$
- Insert $\mathbf{P}$ to get $c = -\mathbf{P}^\top \mathbf{n} \implies$ general normal form
- Normalize to get normal form ("Hessische Normalform")

$$\frac{n_1}{\sqrt{n_1^2 + n_2^2}}\, x_1 + \frac{n_2}{\sqrt{n_1^2 + n_2^2}}\, x_2 + \frac{c}{\sqrt{n_1^2 + n_2^2}} = 0$$

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New Algorithm: Parametric $\rightarrow$ Normal Form

- Rearrange $\mathbf{X} = \mathbf{P} + \lambda \mathbf{v}$ to $\begin{pmatrix} -v_1 & 1 & 0 \\ -v_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$
- Use Givensreduction (one rotation) to eliminate parameter λ

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \Longrightarrow \begin{pmatrix} -cv_1 + sv_2 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} \lambda \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}$$

- Normal form is $s\, x_1 + c\, x_2 - p_2' = 0$

```
function g=Para2Normal2D(P,v)
% PARA2NORMAL2D converts a straight line in parameter form
%     X=P+lambda v to normal form g=[n_1,n_2,c] with X'n+c=0.
B=[-v, eye(2)];
[B,b]=GivensReduction(B,P);
g=[B(2,2:3), -b(2)];
```

Plane: Parametric to Normal Form, Traditional Approach

- $\mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s} \quad \rightarrow \quad g : n_1x_1 + n_2x_2 + n_3x_3 + c = 0$
- $\mathbf{n} = \mathbf{r} \times \mathbf{s}$ cross product
- insert point $\mathbf{P}$ to get c

```
function g=Par2Normal3DConv(P,r,s)
% PAR2NORMALCONV converts the plane X=P+lambda r + mu s to
%    normal form  g=[n', c] where  X'n+c=0 with ||n||=1.

n=kreuz(r,s); n=n(:);       % normal
c=-P'*n;                    % insert P
g=[n',c];
g=g/norm(n);                % normal form
end

function n=kreuz(u,v);
% KREUZ computes the cross product of u and v.
%  corresponds to Matlab function cross
n(1)=u(2)*v(3)-u(3)*v(2);
n(2)=u(3)*v(1)-u(1)*v(3);
n(3)=u(1)*v(2)-u(2)*v(1);
n=n(:);
end
```

New Algorithm: Plane Parametric to Normal Form

- Rearrange plane equations $\mathbf{X} = \mathbf{A} + \lambda \mathbf{r} + \mu \mathbf{s}$ (3 equations, 5 unknowns)

$$\begin{pmatrix} -r_1 & -s_1 & 1 & 0 & 0 \\ -r_2 & -s_2 & 0 & 1 & 0 \\ -r_3 & -s_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

- Eliminate λ and μ using Givensreduction (3 rotations)
- The remaining third equation is the normal form (Hessische Normalform).

```
function g=Par2Normal3D(A,r,s)
% PAR2NORMAL converts the plane X=A+lambda r + mu s to
%    normal form  g=[n', c] where  X'n+c=0 with ||n||=1.
C=[-r, -s, eye(3)];          % form 3x5 matrix
[B,b]=GivensReduction(C,A);  % reduce system
g=[B(3,3:5), -b(3)];         % read the coefficients
```

Plane: from Normal to Parametric Form

$$E : n_1x_1 + n_2x_2 + n_3x_3 + c = 0 \quad \rightarrow \quad \mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s}$$

- choose 3 points A, B and C on E in general position

  $$\Longrightarrow \mathbf{X} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) + \mu(\mathbf{C} - \mathbf{A})$$

  Good for computing by hand.

- For foolproof program make sure $\mathbf{r}, \mathbf{s} \perp \mathbf{n}$, and linear independent

Case 1: $n_k \neq 0, \; k = 1, 2,3$

$$\mathbf{r} = \begin{pmatrix} 0 \\ n_3 \\ -n_2 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} n_3 \\ 0 \\ -n_1 \end{pmatrix}.$$

Case 2: $n_1 = 0$

$$\mathbf{r} = \begin{pmatrix} 0 \\ n_3 \\ -n_2 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Case 3: $n_2 = 0$

$$\mathbf{r} = \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Case 4: $n_3 = 0$

$$\mathbf{r} = \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Conversion from Normal to Parametric Form(cont.)

```
function [P,r,s]=Norm2Para3D(g)
% NORM2PARA3D computes a paramertric from of a plane from the normal form
n=g(1:3);
if abs(n(1))<1e-10                  % construct linear independent
   r=[0;n(3);-n(2)]; s=[1;0;0];    % direction vectors orth. to normal
elseif abs(n(2))<1e-10
   r=[-n(3);0;n(1)]; s=[0;1;0];
elseif abs(n(3))<1e-10
   r=[-n(2);n(1);0]; s=[0;0;1];
else
   r=[0;n(3);-n(2)]; s=[n(3);0;-n(1)];
end

if abs(n(1))>1e-10                  % choose point P on plane
   P=[-g(4)/n(1);0;0];
elseif abs(n(2))>1e-10
   P=[0;-g(4)/n(2);0];
else
   P=[0;0;-g(4)/n(3)];
end
```

Distance Point to Plane, Traditional Solution

- Given point $\mathbf{Q}$ and plane $g: \mathbf{X} = \mathbf{P} + \lambda \mathbf{r} + \mu \mathbf{s}$
- Convert parametric to normal form
  - $\mathbf{n} = \mathbf{r} \times \mathbf{s}$ cross product
  - $\mathbf{n} = \mathbf{n} / \|\mathbf{n}\|$ normalize
  - insert $\mathbf{P} \implies c = -\mathbf{P}^\top \mathbf{n}$

  $\implies$ normal form $n_1 x_1 + n_2 x_2 + n_3 x_3 + c = 0$
- Distance $d = |\mathbf{Q}^\top \mathbf{n} + c|$
- Intersect straight line $\mathbf{X} = \mathbf{Q} + \lambda \mathbf{n}$ with plane gives projected point $\mathbf{Q}'$.

```
function [d,Qp]=PointPlane3DConv(Q,P,r,s)
% POINTPLANE3DCONV computes the distance and the projected point Qp
%   of the point Q from the plane  X=P+lambda r + mu s.
%   Traditional algorithm.
n=kreuz(r,s);                 % normal vector
n=n(:); n=n/norm(n)           % normalize
c=-P'*n;                      % insert P, get normal form of plane
d=abs(n'*Q+c);                % distance
lambda=-c-n'*Q;               % compute intersection point
Qp=Q+lambda*n;
end
function n=kreuz(u,v);
% KREUZ computes the cross product of u and v.
%  One can also use the Matlab function cross
n(1)=u(2)*v(3)-u(3)*v(2);
n(2)=u(3)*v(1)-u(1)*v(3);
n(3)=u(1)*v(2)-u(2)*v(1);
end
```

Distance Point to Plane, New Algorithm

- Given point $\mathbf{Q}$ and plane $g: \ \mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s}$
- $\mathbf{Q}$ on g? $\iff \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \\ r_3 & s_3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}.$
- Givensreduction $\Longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$
- $a_{22} \neq 0$ (direction vectors are not parallel)
- $c_3 = 0$: Q is on the plane.
- $c_3 \neq 0$: $d = |c_3|$ is distance, λ and μ give projected point Q'

```
function [d,Qp]=PointPlane3D(Q,P,r,s)
% POINTPLANE3D computes the distance and the projected point Qp
%   of the point Q from the plane  X=P+lambda r + mu s.
A=[r s]; c=(Q-P);                % lin. system for lambda and mu
[B,b]=GivensReduction(A,c);
mue=b(2)/B(2,2);                 % backsubstitution
lambda=(b(1)-B(1,2)*mue)/B(1,1);
Qp=P+lambda*r+mue*s;             % projected point
d=abs(b(3)); if d<1e-10, d=0;end
```

Plane and Straight Line

- Given plane $E : \mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s}$ and straight line $g : \mathbf{X} = \mathbf{Q} + \rho\mathbf{u}$
- Intersection? $\implies \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s} = \mathbf{Q} + \rho\mathbf{u}$

$$\iff \begin{pmatrix} r_1 & s_1 & -u_1 \\ r_2 & s_2 & -u_2 \\ r_3 & s_2 & -u_2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \rho \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$

- Givensreduction

$$\implies \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \rho \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad a_{11} \neq 0, a_{22} \neq 0$$

- $a_{33} \neq 0$: $\rho = c_3/a_{33}$, intersection point $R = \mathbf{Q} + \rho\mathbf{u}$
- $a_{33} = 0$: line g parallel to E, distance $d = |c_3|$
- Traditional Algorithm with Gaussian Elimination: no distance!

Two Planes $E_1 : \mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s} \qquad E_2 : \mathbf{X} = \mathbf{Q} + \rho\mathbf{u} + \nu\mathbf{v}.$

- Intersection: $\mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s} = \mathbf{Q} + \rho\mathbf{u} + \nu\mathbf{v}$

  $\iff \mathbf{r}\lambda + \mathbf{s}\mu - \mathbf{u}\rho - \mathbf{v}\nu = \mathbf{Q} - \mathbf{P}$

- Givens reduction

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \rho \\ \nu \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad a_{11} \neq 0, a_{22} \neq 0$$

- $a_{33} \neq 0 \implies \rho = \dfrac{c_3}{a_{33}} - \dfrac{a_{34}}{a_{33}}\nu$, insert in E_2

  intersection line $\mathbf{X} = \left(\mathbf{Q} + \dfrac{c_3}{a_{33}}\mathbf{u}\right) + \nu\left(\mathbf{v} - \dfrac{a_{34}}{a_{33}}\mathbf{u}\right)$

- $a_{33} = 0$: $\implies$ $\mathbf{u}$ is linear dependent of $\mathbf{r}$ and $\mathbf{s}$

  If $a_{34} \neq 0 \implies \nu = \dfrac{c_3}{a_{34}}$, intersection line $\mathbf{X} = \left(\mathbf{Q} + \dfrac{c_3}{a_{34}}\mathbf{v}\right) + \rho\mathbf{u}$

  If $a_{34} = 0 \implies E_1$ parallel E_2, distance $d = |c_3|$

```
function [S,w,d]=TwoPlanesPara3D(P,r,s,Q,u,v)
% TWOPLANESPARA3D computes the intersection line X=S+sigma w
% of the two planes  E1=P+lamda r + mue s and E2=Q+rho u + nu v.
% If the planes are parallel the distance d is computed
A=[r, s, -u, -v]; c=[Q-P];
[A,c]=GivensReduction(A,c);
if abs(A(3,3))>1e-10           % normal case
  d=[];
  S=Q+c(3)/A(3,3)*u;
  w=v-A(3,4)/A(3,3)*u ;
elseif abs(A(3,4))>1e-10       % solve for nu
  d=[];
  S=Q+c(3)/A(3,4)*v;
  w=u;
else                           % planes are parallel
  d=abs(c(3)); S=[];w=[];
end
```

Two Planes, Traditional Approach

$E_1 : \mathbf{X} = \mathbf{P} + \lambda\mathbf{r} + \mu\mathbf{s} \qquad E_2 : \mathbf{X} = \mathbf{Q} + \rho\mathbf{u} + \nu\mathbf{v}.$

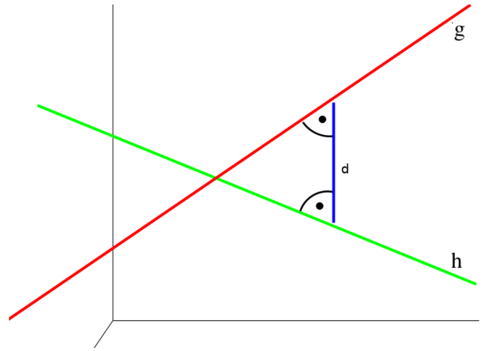
1. Convert the planes to normal form, normal vectors $\mathbf{n}_{E1}$ and $\mathbf{n}_{E2}$
2. If $\mathbf{n}_{E1}$ and $\mathbf{n}_{E2}$ not parallel $\implies \exists$ intersection line.
   - Find general solution of normal equations of E_1 and E_2
   - Or find common point on E_1 and E_2 and $n = \mathbf{n}_{E1} \times \mathbf{n}_{E2}$
3. If $\mathbf{n}_{E1}$ and $\mathbf{n}_{E2}$ parallel, check if normal equations are multiples of each other:
   (a) if multiple $\implies$ E_1 and E_2 are coincident
   (b) no multiple $\implies$ E_1 and E_2 are parallel
   (c) distance? compute e.g. distance of $\mathbf{P}$ from E_2

Source: abimatura | Mathe lernen
`https://www.abiturma.de/mathe-lernen/geometrie/lagebeziehungen-und-schnitt/lagebeziehung-ebene-ebene`

Two straight Lines in Space $g : \mathbf{X} = \mathbf{P} + \lambda \mathbf{r}, \qquad h : \mathbf{X} = \mathbf{Q} + \mu \mathbf{s}.$

- Intersection? $\mathbf{P} + \lambda \mathbf{r} = \mathbf{Q} + \mu \mathbf{s}$

$$\begin{pmatrix} r_1 & -s_1 \\ r_2 & -s_2 \\ r_3 & -s_3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$



- Givensreduction $\Longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$

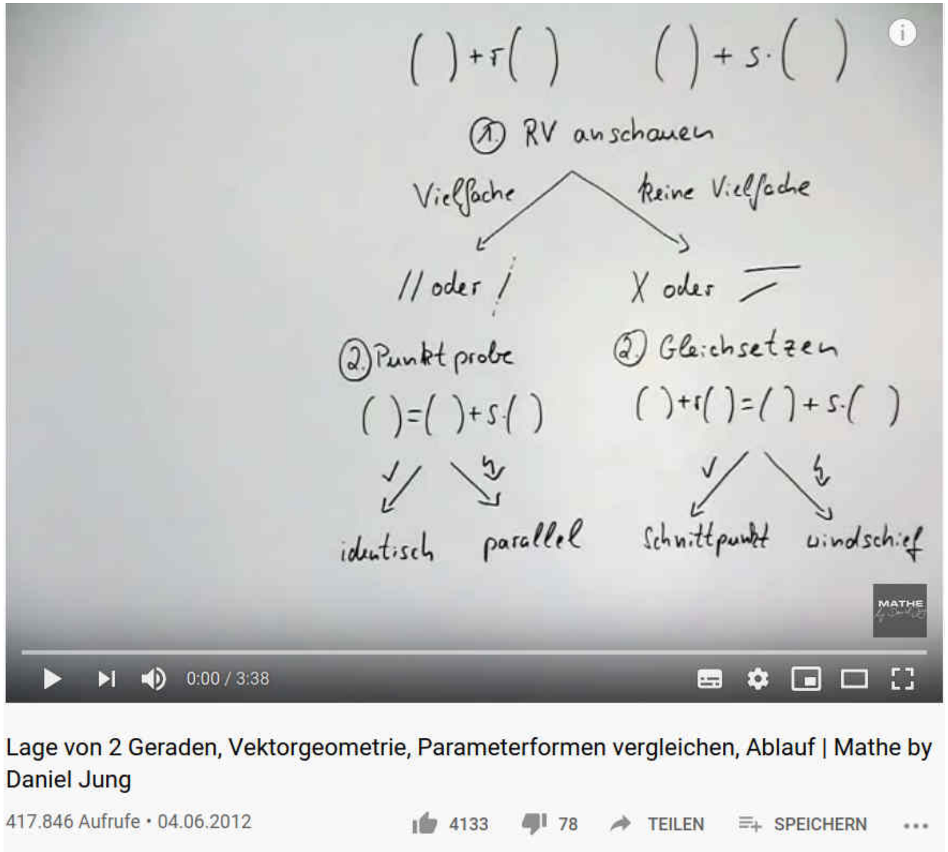
- Interpretation:
  - If $a_{22} \neq 0$ compute λ and μ. Intersection point if $c_3 = 0$.
    If $c_3 \neq 0$, lines skewed, λ and μ give nearest points, distance $d = |c_3|$.
  - If $a_{22} = 0 \quad \Longrightarrow$ lines parallel. Distance $d = \sqrt{c_2^2 + c_3^2}$.

```
function [d,R,S,Ac]= TwoLines3D(P,r,Q,s)
% TWOLINES3D computes the position of the straight lines
%   X=P+lambda r and X=Q+nu s in space.
%   R,S are the closest points on the lines
%   d distance, d=||R-S||
%   Ac is matrix of the reduced system together with right hand side c
%
A=[r,-s]; c=[Q-P];              % equations for lambda and mu
[A,c]=GivensReduction(A,c);
Ac = [A, c];                    % reduced system
if abs(A(2,2))>1e-10            % can solve for lambda and mu
    mu=c(2)/A(2,2)
    lambda=(c(1)-A(1,2)*mu)/A(1,1)
    R=P+lambda*r; S=Q+mu*s;
    d=abs(c(3));                % d is equal to norm(R-S)
else                            % direction vectors are parallel
    d=norm(c(2:3));             % distance between parallel lines
    R=[]; S=[];
end
```

Two straight Lines, Traditional $g : \mathbf{X} = \mathbf{P} + \lambda\mathbf{r}, \quad h : \mathbf{X} = \mathbf{Q} + \mu\mathbf{s}.$

Compare direction vectors $\mathbf{r}$ and $\mathbf{s}$

1. if multiples $\Longrightarrow$ g and h are parallel or coincident
   - point $\mathbf{P}$ on h: coincident
   - point $\mathbf{P}$ not on h: parallel
2. $\mathbf{r}$ and $\mathbf{s}$ not multiples $\Longrightarrow$ intersection or warp
3. solve $\mathbf{P} + \lambda\mathbf{r} = \mathbf{Q} + \mu\mathbf{s}$ and decide if intersect.



https://www.youtube.com/watch?v=GYbf-kCRJJI